Introduction to Solitons

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Electromagnetic duality and Dirac monopole

- System of generalized Maxwell equations

\[ \nabla \cdot \vec{E} = 4\pi e; \quad \nabla \cdot \vec{B} = 4\pi g; \]
\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{j}_g \]
\[ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}_e \]

is invariant with respect to the transformations of electromagnetic duality:

\[ E \rightarrow E \cos \vartheta - B \cos \vartheta; \quad e \rightarrow e \cos \vartheta - g \cos \vartheta; \]
\[ B \rightarrow E \sin \vartheta + B \cos \vartheta \]
\[ g \rightarrow e \sin \vartheta + g \cos \vartheta \]

- Classical motion in the monopole Coulomb magnetic field:

\[ m \frac{d^2 \vec{r}}{dt^2} = e [\vartheta \times \vec{B}] = \frac{eg}{r^3} \left[ \frac{d\vec{r}}{dt} \times \vec{r} \right] \]

- Generalized angular momentum is conserved:

\[ \vec{J} = [\vec{r} \times m\vec{v}] - e g \frac{\vec{r}}{r} = \vec{L} - eg\hat{r} \]
Dirac’s monopole: Charge quantization

\[ \vec{B} = g \frac{\vec{r}}{r^3} = \nabla \times \vec{A}; \quad \nabla \cdot \vec{B} = 4\pi g \ ? \]

\[ \vec{A} = \frac{g}{r} \frac{\vec{r} \times \vec{n}}{r - (\vec{r} \cdot \vec{n})} \quad \text{- Dirac monopole potential} \]

\[ \vec{B} = \vec{B}_g + \vec{B}_{\text{string}} = g \frac{\vec{r}}{r^3} - 4\pi g \ \vec{n} \ \theta(z)\delta(x)\delta(y) \]

Gauge invariance: \[ \vec{A} \rightarrow \vec{A} + \nabla \lambda(\vec{n}, \vec{n}') \]

*Dirac’s string is invisible if the charge quantization condition is imposed:*

\[ eg = \frac{n}{2} \]
Non-Abelian monopoles

Break-through of 1974: *‘t Hooft-Polyakov monopole solution*

While a Dirac monopole could be incorporated in an Abelian theory, some non-Abelian models inevitably contain monopole solutions.

Non-Abelian monopole is a non-linear system of coupled gauge and scalar (Higgs) fields, its energy is finite and the fields are regular everywhere in space. The gauge symmetry is spontaneously broken via Higgs mechanism.

\[
L = \frac{1}{2} \text{Tr } F_{\mu\nu}^2 + \text{Tr } (D_\mu \Phi)^2 + V(|\Phi|)
\]
Magnetic monopoles

\[ \vec{B} = g \frac{\vec{r}}{r^3}, \quad \vec{E} = Q \frac{\vec{r}}{r^3} \]

**Dirac monopole (1931)**

**Non-Abelian monopole**

**Wu-Yang monopole (1975)**

\[ \begin{align*}
A_N^\theta &= g \frac{1 - \cos \theta}{r \sin \theta} \hat{e} \\
A_S^\theta &= -g \frac{1 + \cos \theta}{r \sin \theta} \hat{e}
\end{align*} \]

- Regular static configuration
- Gauge group SU(2)
- Magnetic charge is the topological number: \( Qg = n/2 \)
- The monopole is very heavy, \( M \sim m_v/e \)
Properties of non-Abelian monopoles [$SU(5)$]

- Monopole has a core of radius $r_m \sim m_x^{-1} \sim 10^{-29}$ cm
- Monopole is superheavy: $M \sim m_x/\alpha \sim 10^{17}$ GeV $\sim 10^{-7}$ g
- Magnetic charge of the monopole has topological roots:

\[ \vec{\Phi} \rightarrow \nu \vec{r} \]
\[ S^2 \rightarrow S^2 \]

- Electromagnetic subgroup is associated with rotations about direction of the Higgs field
- Monopole solution mixes the spacial and group rotations:

\[ \vec{J} = \vec{L} + \vec{T} + \vec{S} \]
Monopole has 4 collective coordinates: \( R_k \) and \( \chi(t) \)

Electric charge of a dyon is \( Q = 4\pi \dot{\chi} \)

Charge quantization condition for a pair of dyons: \( Q_1 g_2 - Q_2 g_1 = \frac{n}{2} \)

Consequence: Spin-statistic theorem admits both Bose-Einstein and Fermi-Dirac statistics.
Monopoles should have been produced in the very early Universe:

\[ SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)_{em} \]

As \( T < T_c \sim 10^{15} \text{ GeV} \) the Higgs field acquires a non-zero v.e.v.

Predictions of the Big Bang scenario (adiabatic expansion)

\( 1 \text{ monopole per } 10^4 \text{ nucleons!} \)

Inflation scenario: The potential is sufficiently flat at \( \Phi = 0 \), the phase transition occurs at \( T_c \sim 10^9 \text{ GeV} \)

- only a few monopoles may survive the inflation!
Experimental search for monopoles

- Accelerator search
  (Fermilab, CERN, DESY…)

- Superconducting coils

- Indirect limits (Parker’s bound, neutron stars…)

- Search for monopole catalysis
  (IceCube, Berkeley, Stanford, IBM…)

- Monopoles in cosmic rays
  (Scintillators and ionization detectors)

No monopole detected yet!

(©Picture by courtesy G. Giacomelli)
Fake monopoles

"Monopoles" in spin-ice crystal structures


\[
H = J \sum_{ij} S_i S_j + \sigma \sum_{ij} \frac{3(\hat{e}_i \cdot \hat{r}_{ij})(\hat{e}_j \cdot \hat{r}_{ij}) - (e_i \cdot e_j)}{r_{ij}^3}
\]

A sum of nearest-neighbor Ising model term and long range dipolar interactions

(Mengotti et al, Nature Physics, 7 (2011) 68)

Where is the cheat?

Each dipole is replaced by a pair of equal and opposite magnetic charges
Fake monopoles

● „Monopoles“ and low energy QCD

$\Lambda_\chi \sim 1 \text{ GeV}$
$\Lambda_{QCD} \sim 180 \text{ MeV}$

Perturbative QCD (Quarks & gluons) Low energy effective theory (Pions and quarks) Hadrons

QCD confinement as dual Meissner effect: monopole condensation as a reason of formation of the chromoelectric flux tube and QCD is taking a form of the dual Ginzburg-Landau model (S. Mandelstam, G ’t Hooft et al (1970s))

Where is the cheat?

*There is no monopoles in QCD!*
Here we are: Yang-Mills-Higgs Theory

\[ S = \frac{1}{2} \int d^4x \left\{ F_{\mu\nu}F^{\mu\nu} + (D_\mu \Phi)(D^\mu \Phi) - V(\Phi) \right\} \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \]
\[ D_\mu \Phi = \partial_\mu \Phi + ie[A_\mu, \Phi] \]
\[ V(\Phi) = \lambda (\Phi^2 - a^2)^2 \]

\'t Hooft-Polyakov static spherically symmetric solution

\[ \phi^a = \frac{r^a}{er^2} H(ear) \]
\[ A^a_n = \varepsilon_{amn} \frac{r^m}{er^2} (1 - K(ear)) \]

Monopole core: \( R_c \sim m_v^{-1} \)
**BPS monopoles**

**Bogomolny equations:** \[ \lambda = 0, \quad B_k = D_k \Phi \]

**BPS monopole mass:** \[ M = 4\pi \eta / e \quad \text{Homotopy group} \quad \pi_2(S^2) \]

- Long-range scalar field \( \Phi \sim 1/r \)
- No net interaction between the BPS monopoles
- Analytical solution of the BPS equations:

\[
K = \frac{\xi}{\sinh \xi}; \quad H = \xi \coth \xi - 1
\]

Magnetic charge of a monopole is a topological number \( \Phi : S^2 \to S^2 \)

Self-dual monopoles vs non-self dual monopoles

**BPS monopoles:**
- In the limit $\eta=0$ the energy becomes:
  \[ E = \text{Tr} \int d^3x \left\{ \frac{1}{4} (\varepsilon_{ijk} F_{ij} \pm D_i \Phi)^2 \mp \frac{1}{2} \varepsilon_{ijk} F_{ij} D_k \Phi \right\} \]
- The first order Bogomol'nyi equations $B_k = \pm D_k \Phi$ yield absolute minimum:
  \[ M = 4\pi g \]
- No net interaction between the BPS monopoles: the electromagnetic repulsion is compensated by the long-range scalar interaction.

**Non self-dual monopoles:**
- They are solutions of the second order Yang-Mills equations: $\partial_{\mu} F_{\mu\nu} = 0$
- $E > M_{BPS}$ even if $g=0$ (deformations of the topologically trivial sector)
- The constituents are non BPS monopoles and/or vortices in a static equilibrium; separation is relatively small, there are no long-range forces
Self-dual monopoles vs non-self dual monopoles

**BPS monopoles:**

- Integrability of the BPS equations: there is a correspondence to the reduced self-duality equations of the Yang-Mills theory:

  \[ D_k \Phi^a \Rightarrow D_k A_0^a \equiv F_{0k}^a \quad \quad B_k^a \Rightarrow {1 \over 2} \varepsilon_{k mn} F_{mn} \equiv \tilde{F}_{0k}^a \equiv F_{0k}^a \]

- Properties of the BPS monopole are completely defined by the Higgs field:

  \[ D_k \Phi^a D_k \Phi^a = {1 \over 2} \partial_k \partial_k |\Phi|^2 \quad \quad E = {1 \over 2} \int d^3 x \partial_k \partial_k |\Phi|^2 = 4 \pi g \]

- An infinite chain of YM instantons along Euclidean time axis

  \[ A_k^a = \varepsilon_{akn} \partial_n \ln \rho + \delta_{ak} \partial_0 \ln \rho; \quad A_0^a = -\partial_a \ln \rho \quad \text{where} \quad \rho = \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (\tau - 2 \pi n)^2} \]

  is equal to the BPS monopole \((z = r + i \tau)\):

  \[ \rho = \frac{1}{2r} \left\{ \sum_{n=-\infty}^{n=\infty} \frac{1}{z - i \omega_n} + \sum_{n=-\infty}^{n=\infty} \frac{1}{z^* + i \omega_n} \right\} = \frac{1}{2r} \left\{ \coth \frac{z}{2} + \coth \frac{z^*}{2} \right\} = \frac{1}{2r} \frac{\sinh r}{\cosh r - \cos \tau}. \]
Rational map monopoles

There is a transformation of a monopole into a rational map from the Riemannian sphere to itself: \( R: \mathbb{S}^2 \rightarrow \mathbb{S}^2 \) (P. Sutcliffe, N. Manton et al)

\[
R(z) = \frac{a(z)}{b(z)} = \frac{a_1 z^{n-1} + \cdots + a_n}{z^n + b_1 z^{n-1} + \cdots + b_n}, \quad z = x_1 + ix_2
\]

Construction of the rational maps monopoles:

- Represent BPS equation in spherical coordinates \( r, z, \bar{z} \)
- Impose a complex gauge \( \Phi = -iA_r = \frac{i}{2}U^{-1}\partial_r U, \quad A_z = U^{-1}\partial_z U, \quad A_{\bar{z}} = 0 \)
- Construct the monopoles using

\[
U \sim \exp \left\{ \frac{2r}{1 + |R|^2} \left( \begin{array}{cc} |R|^2 - 1 & 2\bar{R} \\ 2R & 1 - |R|^2 \end{array} \right) \right\}
\]
\( R = 1/z: \) one spherically symmetric monopole centered at the origin;

\[
R(z) = \frac{a_1 z + a_2}{z^2 + b_1 z + b_2} : \text{two monopoles}
\]

\[
R(z) = \frac{i\sqrt{3}z^2 - 1}{z(z^2 - i\sqrt{3})} : \text{Tetrahedral monopoles (degree 3 map)}
\]

\[
R(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2iz^2\sqrt{3} + 1} : \text{Octahedral monopoles (degree 4 map)}
\]
Monopole catalysis of proton decay

**Naive question:** What happened when a fermion collides with a monopole?

\[ \bullet \rightarrow \begin{array}{c} \text{fermion} \\ \end{array} = \begin{array}{c} \text{monopole} \\ \text{add?} \end{array} \]

There are zero-energy solutions of the Dirac equation for a massless fermion coupled to a monopole:

\[ \gamma^\mu (\partial_\mu + eA_\mu)\psi = 0; \quad \vec{J} = \vec{L} + \vec{T} + \vec{S}; \quad \vec{L} = 0, \quad \vec{T} + \vec{S} = 0 \]

The ground state of a monopole becomes two-fold degenerated:

(i) \[ |\Omega\rangle \quad \text{(no fermions; } Q_F = -1/2) \]

(ii) \[ a^\dagger |\Omega\rangle \quad \text{(zero mode; } Q_F = 1/2) \]

**Spin-flip? Charge conjugation? Chirality?**

There are non-suppressed fermionic condensates on the monopole background:

\[ \langle (e^+ e^- - d^3 \bar{d}^3)(\bar{u}^1 u^1 - \bar{u}^2 u^2) \rangle \sim r^{-6} \]
Rubakov-Callan effect

SU(5) model with massless fermions in s-wave

Monopole could catalyse baryon number violating processes like

\[ P + \text{monopole} \rightarrow e^+ + \text{mesons} + \text{monopole} \]

**Caution:** the effect is model-dependent!
Non-Abelian monopoles and black holes
Gravitating monopoles

\[
\frac{1}{\sqrt{-\text{g}}} D_\mu (\sqrt{-\text{g}} F^{\mu \nu}) - \frac{1}{4} i e [\Phi, D^\nu \Phi] = 0; \\
\frac{1}{\sqrt{-\text{g}}} D_\mu (\sqrt{-\text{g}} D^{\mu} \Phi) + \lambda (\Phi^2 - a^2) \Phi = 0.
\]

(Cho, Freund (1975), van Nieuwenhuizen et al (1976), Breitenlohner, Forgacs, Maison (1992), Lee, Nair, Weinberg (1992), Hartmann, Kleihaus Kunz, Shnir…)

Dimensionless parameters of the model: \(a^2 \equiv 4\pi^2 \text{G}\eta^2, \ \beta^2 \equiv e^2/\eta\)

Monopole core \(R_c \sim m_\nu^{-1} = (e\eta)^{-1}\) vs Schwarzschild radius \(R_{\text{Sch}} = 2MG;\)

\[R_c \sim R_{\text{Sch}} \text{ as } \eta \sim M_{\text{Pl}} = \text{G}^{-1/2}\]

Hairy black holes with axial symmetry are linked to monopoles
Global monopoles (monopole as big as Universe)

\[ L = \frac{1}{2} (\partial_{\mu} \Phi^{a})^2 - V(|\Phi|); \quad SO(3) \rightarrow O(2) \]

There is no vector (gauge) field - but gravity may be coupled to this system instead

**Topological defects & extra dimensions:**

**Our D=4 world:** the internal space of a topological defect living in a higher dimensional space-time

- **d=5:** domain wall (kink in d=1 + D=4)
- **d=6:** vortex (in d=2) + D=4
- **d=7:** monopole (in d=3) + D=4
- **d=8:** instanton (in d=4) + D=4
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Work in progress!

“At this point we notice that this equation is beautifully simplified if we assume that space-time has 99 dimensions.”